A GEOMETRIC INTERPRETATION AND A NEW PROOF OF A RELATION BY CORNALBA AND HARRIS

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Abstract. In the 80's M. Cornalba and J. Harris discovered a relation among the Hodge class and the boundary classes in the Picard group with rational coefficients of the moduli space of stable, hyperelliptic curves. They proved the relation by computing degrees of the classes involved for suitable one-parameter families. In the present article we show that their relation can be obtained as the class of an appropriate, geometrically meaningful empty set, thus conforming with C. Faber's general philosophy to finding relations among tautological classes in the Chow ring of the moduli space of curves. The empty set we consider is the closure of the locus of smooth, hyperelliptic curves having a special ramification point.

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1. Introduction

1.1. *Mumford's relation and Faber's idea.* Let $\overline{M}_g$ be the moduli space of (Deligne-Mumford) stable curves of genus g, and $\mathrm{Pic}(\overline{M}_g)$ its Picard group; see [5] and [14]. Natural classes of $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ are the Hodge class λ and the boundary classes $\delta_0, \ldots, \delta_{[g/2]}$; see [1]. It is a fundamental result by J. Harer, E. Arbarello and M. Cornalba that the above classes freely generate $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ for each $g \geq 3$; see [13] and [1]. However, for $g = 2$ there is the relation below, proved by D. Mumford in his fundamental paper [17],

$$10\lambda - \delta_0 - 2\delta_1 = 0. \tag{1}$$

Mumford's relation (1) was recovered by S. Diaz and F. Cukierman in [6] and [4], though they do not observe it there. In fact, Diaz considers the locus $D_{g-1,g-1} \subseteq \overline{M}_g$ of smooth curves having a (Weierstrass) point whose first non-zero non-gap is at most $g-1$, and computes the class of its closure in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ ([6], Thm. (7.4), p. 40). Similarly, Cukierman computes the class of the closure of the locus $\mathcal{E} \subseteq \overline{M}_g$ of smooth curves having a global holomorphic

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1-form vanishing at a (Weierstrass) point with order at least $g+1$ ([4], Eq. (5.5), p. 344 and Rmk. (b) thereafter). Both Diaz and Cukierman claim their formulas hold for $g \geq 3$. However, we observed that their formulas hold for $g=2$ as well, and give Mumford's relation (1). Indeed, for $g=2$ both $D_{1,1}$ and $\mathcal{E}$ are empty, and so the formulas given by Diaz and Cukierman express the class of the empty set, 0, as a linear combination of λ, δ_0 and δ_1. In both cases, this linear combination is a non-zero multiple of that in (1).

Therefore, Mumford's relation (1) fits perfectly in C. Faber's philosophy to finding relations among tautological classes in Chow rings of moduli spaces of curves by writing down classes of "appropriate" empty sets; see [10]. The beauty in Faber's idea is that relations discovered by his approach are endowed with geometric interpretations.

Now, let $H_g \subseteq \overline{M}_g$ be the locus of smooth, hyperelliptic curves, and $\overline{H}_g \subseteq \overline{M}_g$ its closure. A stable curve represented in $\overline{H}_g$ is also called hyperelliptic. Of course, each smooth curve of genus 2 is hyperelliptic, hence $\overline{H}_2 = \overline{M}_2$. So we may view Mumford's relation as one in $\operatorname{Pic}(\overline{H}_2) \otimes \mathbb{Q}$. In general, M. Cornalba and J. Harris proved in [3] the theorem below; see also Subsection 6C of [14].

Theorem 1.2. (Cornalba-Harris) *The following relation holds in* $\operatorname{Pic}(\overline{H}_g) \otimes \mathbb{Q}$*:*

$$(8g+4)\lambda - g\xi_0 - \sum_{i=1}^{[(g-1)/2]} 2(i+1)(g-i)\xi_i - \sum_{j=1}^{[g/2]} 4j(g-j)\delta_j = 0. \tag{2}$$

In the above statement, $\xi_0, \dots, \xi_{[(g-1)/2]}$ are boundary classes satisfying the relation below; see Sections 3 and 4.

$$\delta_0 - \xi_0 - 2(\xi_1 + \cdots + \xi_{[(g-1)/2]}) = 0. \tag{3}$$

For $g=2$, Cornalba's and Harris' relation (2) coincides with Mumford's (1).

Cornalba and Harris found (2) by computing the degree of λ for one-parameter families of admissible covers. Now, our observation above suggests that, like Mumford's, also Cornalba's and Harris' relation might be obtained by expressing the class of an "appropriate" empty set, thus shedding light on its geometric significance. As in the works by Diaz and Cukierman, the "appropriate" empty set might be the locus of curves having a certain type of Weierstrass point. Indeed, the goal of the present article is to show that *a non-zero multiple of the left-hand side of* (2) *expresses the class in* $\operatorname{Pic}(\overline{H}_g) \otimes \mathbb{Q}$ *of the closure* $\overline{W}_{g,2}$ *of the locus* $W_{g,2} \subseteq \overline{H}_g$ *of smooth, hyperelliptic curves for which the unique* g^1_2 *has a special ramification point.*

1.3. *An overview.* As Cornalba and Harris in [3], we will rather work with the moduli stack $\overline{\mathcal{H}}_g$ of stable, hyperelliptic curves of genus g, and show relation (2) in $\operatorname{Pic}(\overline{\mathcal{H}}_g) \otimes \mathbb{Q}$. In other words, we will consider a family $\pi\colon C \to S$ of stable, hyperelliptic curves and show that the induced classes λ_π, $\xi_{1,\pi}, \dots, \xi_{[(g-1)/2],\pi}$ and $\delta_{1,\pi}, \dots, \delta_{[g/2],\pi}$ in $\operatorname{Pic}(S) \otimes \mathbb{Q}$ satisfy the relation

$$(8g+4)\lambda_\pi - g\xi_{0,\pi} - \sum_{i=1}^{[(g-1)/2]} 2(i+1)(g-i)\xi_{i,\pi} - \sum_{j=1}^{[g/2]} 4j(g-j)\delta_{j,\pi} = 0. \tag{4}$$

We will see that it is sufficient to consider a particular family $\pi\colon C \to S$ where S is a smooth, projective curve, the general fiber of π is smooth and admits a g^1_2, and the singular fibers are "general enough"; see Section 4. So let us assume π is this family.

Let $\widetilde{\pi}\colon \widetilde{C} \to S$ be the semi-stable reduction of π. In Section 5 we consider the wronskian determinant w of a certain relative g^1_2 on $\widetilde{\pi}$, and use the theory of limit linear series developed in [9] to compute the orders of vanishing of w on the irreducible components of the singular fibers of $\widetilde{\pi}$. Subtracting these irreducible components with the computed multiplicities, we obtain a section $\overline{w}$ of a line bundle on $\widetilde{C}$ cutting out a relative Cartier divisor $\overline{W}$ over S. Now, as in [12] (see also [11]), we consider in Section 6 the degeneracy locus Z of the derivative $D\overline{w}$ of $\overline{w}$. This locus is empty, as we show that $\overline{W}$ is unramified over S for the particular family π we consider. Since Z has the expected dimension, the class of $[Z]$ coincides with the degeneracy class of $D\overline{w}$, what allows us to compute an expression for $\widetilde{\pi}_*[Z]$ in terms of λ_π, $\xi_{1,\pi}, \dots, \xi_{[(g-1)/2],\pi}$ and $\delta_{1,\pi}, \dots, \delta_{[g/2],\pi}$. This expression is the class in $\mathrm{Pic}(S) \otimes \mathbb{Q}$ of the closure of the set of points $s \in S$ for which the fiber C_s is smooth and has a special ramification point for its unique g^1_2. This set is empty, and the expression is a multiple of the left-hand side of (4). Thus (4) is proved.

In Section 2 we exemplify our method by showing (1). This is simpler than showing (2), because all the singular curves we have to deal with are either irreducible or of compact type, and hence we may apply results by Diaz and Cukierman in [6] and [4]. So, not only are the technical tools kept to a minimum, but also are available in the literature. To conform to the technical tools available, the proof in Section 2 is just slightly different from the one laid out above.

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2. Mumford's relation

2.1. *Reduction.* Let $\pi\colon C \to S$ be a family of stable curves of genus g. Denote by ω_π its relative dualizing sheaf. The direct image $E_\pi := \pi_*\omega_\pi$ is called the *Hodge bundle* of π. Let $\lambda_\pi := c_1(E_\pi)$ in $\mathrm{Pic}(S)$. For each invertible sheaf $\mathcal{L}$ on C, let $J^1_\pi(\mathcal{L})$ be the *sheaf of relative jets*, or principal parts of order 1 of $\mathcal{L}$; see Section 2 of [11] or [8]. As for families of smooth curves, there is a natural exact sequence,

$$0 \to \mathcal{L} \otimes \omega_\pi \to J^1_\pi(\mathcal{L}) \to \mathcal{L} \to 0. \tag{5}$$

If S and the general fiber of π are smooth curves then the singularities of the total space C of π are concentrated at the nodes of the singular fibers and are mild. More precisely, the surface C has at each node P of each singular fiber of π a singularity of type A_k for a certain integer k, called the *singularity type of P in C*. Then C is smooth if and only if the singularity type in C of every node of every singular fiber of π is 1.

Using level structures, one can show that there is a family of stable curves over a smooth base Ω represented by a finite and surjective map $\Omega \to \overline{M}_g$; see [15] or [2]. Hence, to show a relation among classes in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$, it is enough to prove that for a "sufficiently general" one-parameter family of stable curves the induced classes satisfy the corresponding relation.

Assume $g = 2$ for the rest of Section 2. Let $\pi\colon C \to S$ be an one-parameter family of stable curves of genus 2. Assume π is "sufficiently general". More precisely, assume S and the general fiber of π are smooth curves, and each of the singular fibers of π is a general

uninodal curve; in particular, each fiber of π has at most two irreducible components. To prove (1) we need only show that the classes $\lambda_\pi, \delta_{0,\pi}, \delta_{1,\pi}$ in $\mathrm{Pic}(S) \otimes \mathbb{Q}$ induced by $\lambda, \delta_0, \delta_1$ satisfy

$$10\lambda_\pi - \delta_{0,\pi} - 2\delta_{1,\pi} = 0. \tag{6}$$

For each $s \in S$ such that C_s is singular, let k_s be the singularity type in C of the unique node of C_s. For each $i = 0, 1$, let $\Delta_{i,\pi}$ be the set of points $s \in S$ for which the fiber C_s is singular with $i+1$ irreducible components. Then $\delta_{i,\pi}$ is the divisor class of points $s \in \Delta_{i,\pi}$ counted each with multiplicity k_s; see Section 3D of [14].

2.2. *Computation.* Let w be the section of the line bundle $\omega_\pi^3 \otimes \pi^* \wedge^2 E_\pi^*$ obtained by taking determinants in the natural map of vector bundles $\pi^* E_\pi \to J_\pi^1(\omega_\pi)$. The zero scheme W of w cuts each smooth fiber of π in the scheme of Weierstrass points of that fiber. Using the local computations by Cukierman for Prop. (2.0.8) on p. 325 of [4], we see that w vanishes along each irreducible component of C_s for each $s \in \Delta_{1,\pi}$ with order k_s. Let $\overline{w}$ be the induced section of $\omega_\pi^3 \otimes \pi^*(\wedge^2 E_\pi^*(-D))$, where $D := \sum_{s \in \Delta_1} k_s[s]$. Let $\overline{W}$ be the zero scheme of $\overline{w}$. Then $\overline{W}$ is a relative Cartier divisor on C over S. By considerations of degree, away from possible nodes the divisor $\overline{W}$ cuts each fiber C_s for each $s \in S$ transversally. By Theorem (A2.4) on page 64 of [6], for each $s \in \Delta_{1,\pi}$ the divisor $\overline{W}$ cuts the fiber C_s away from the node. In contrast, by Theorem (A2.1) on page 60 of [6], the divisor $\overline{W}$ cuts the fiber C_s for each $s \in \Delta_{0,\pi}$ with multiplicity 2 at the node.

Let Z be the degeneracy locus of the derivative $D\overline{w}$ of $\overline{w}$. Here $D\overline{w}$ is the section of

$$J_\pi^1(\omega_\pi^3 \otimes \pi^*(\wedge^2 E_\pi^*(-D))) \tag{7}$$

induced by $\overline{w}$ (see [11] or [12] for this construction). Since $\overline{W}$ is a Cartier divisor cutting the fiber C_s for each $s \in \Delta_{0,\pi}$ with multiplicity 2 at the node, a local computation shows that the length of Z at the node is k_s. As Z is supported in the union of these nodes, we have $\pi_*[Z] = \delta_{0,\pi}$. Now, as the degeneracy locus of $D\overline{w}$ has the expected dimension zero, the class of $[Z]$ is the second Chern class of the bundle in (7). So

$$\begin{aligned} \pi_*[Z] =& \pi_*[(3c_1(\omega_\pi) - \pi^*(\lambda_\pi + \delta_{\pi,1}))(4c_1(\omega_\pi) - \pi^*(\lambda_\pi + \delta_{\pi,1}))] \\ =& \pi_*[12c_1(\omega_\pi)^2 - 7c_1(\omega_\pi)\pi^*(\lambda_\pi + \delta_{1,\pi})]. \end{aligned}$$

By Grothendieck-Riemann-Roch, $\pi_*(c_1(\omega_\pi)^2) = 12\lambda_\pi - \delta_{0,\pi} - \delta_{1,\pi}$; see Section 3E of [14] or Theorem 5.10 on page 100 of [16]. By the projection formula, $\pi_*(c_1(\omega_\pi)\pi^*\mu) = 2\mu$ for every $\mu \in \mathrm{Pic}(S)$. Hence,

$$\pi_*[Z] = 12(12\lambda_\pi - \delta_{0,\pi} - \delta_{1,\pi}) - 14(\lambda_\pi + \delta_{1,\pi}) = 130\lambda_\pi - 12\delta_{0,\pi} - 26\delta_{\pi,1},$$

and thus the difference $\pi_*[Z] - \delta_{0,\pi}$ is a multiple of the left-hand side of (6). Since $\pi_*[Z] = \delta_{0,\pi}$ we get (6), thus proving Mumford's relation (1).

3. The Picard group

3.1. *The Picard group of $\overline{M}_g$.* (See [1] or [14].) Let $\Delta_0 \subseteq \overline{M}_g$ be the closure of the locus of irreducible, uninodal, stable curves. For each $i = 1, \ldots, [g/2]$, let $\Delta_i \subseteq \overline{M}_g$ be the closure of

the locus of uninodal, stable curves with two irreducible components, one of them of genus i. Then $\Delta_0, \ldots, \Delta_{[g/2]}$ are divisors of $\overline{M}_g$ covering $\overline{M}_g - M_g$.

Let $\overline{\mathcal{M}}_g$ be the moduli stack of stable curves and $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ its group of divisor classes. Each divisor class on $\overline{M}_g$ defines, by pullback, one on $\overline{\mathcal{M}}_g$. So, there is a well-defined homomorphism $\Psi \colon \mathrm{Pic}(\overline{M}_g) \to \mathrm{Pic}(\overline{\mathcal{M}}_g)$. Moreover, one can show that $\Psi \otimes 1_{\mathbb{Q}}$ is an isomorphism.

There is a class $\lambda \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ whose pullback to $\mathrm{Pic}(S)$ for each family $\pi \colon C \to S$ of stable curves is the class λ_π defined in Subsection 2.1. We call λ the *Hodge class*. In addition, as the universal deformation space of a stable curve is smooth, for each $i = 0, \ldots, [g/2]$ the divisor Δ_i induces a divisor class δ_i in $\mathrm{Pic}(\overline{\mathcal{M}}_g)$. We call $\delta_0, \ldots, \delta_{[g/2]}$ the *boundary classes*. Denote also by $\lambda, \delta_0, \ldots \delta_{[g/2]}$ the corresponding classes in $\mathrm{Pic}(\overline{M}_g) \otimes \mathbb{Q}$ under the isomorphism $\Psi \otimes 1_{\mathbb{Q}}$.

3.2. *The Picard group of $\overline{H}_g$.* (See [3] or Section 6C of [14].) The locus $\overline{H}_g$ is the quotient of the Hurwitz scheme of admissible covers of degree 2 by the action of the symmetric group on $2g+2$ letters. Moreover, the latter scheme is the same thing as the moduli space $\overline{M}_{0,2g+2}$ of stable $(2g+2)$-pointed rational curves. The moduli space $\overline{M}_{0,2g+2}$ is fine, smooth and projective.

The above description of $\overline{H}_g$ implies that $\overline{H}_g - H_g$ has exactly g irreducible components $\Xi_0, \ldots, \Xi_{[(g-1)/2]}$ and $\Theta_1, \ldots, \Theta_{[g/2]}$, all of them divisors of $\overline{H}_g$, given as follows:

1. The component Ξ_0 is the closure of the locus of curves obtained from a smooth, hyperelliptic curve of genus $g-1$ by identifying two distinct points conjugated by the hyperelliptic involution.
2. For each $i = 0, \ldots, [(g-1)/2]$, the component Ξ_i is the closure of the locus of curves with just two irreducible components of genera i and $g-1-i$ which are smooth, hyperelliptic, and intersect transversally at just two distinct points conjugated by the hyperelliptic involution on each curve.
3. For each $j = 1, \ldots, [g/2]$, the component Θ_j is the closure of the locus of curves with just two irreducible components of genera j and $g-j$ which are smooth, hyperelliptic, and intersect transversally at a unique point fixed by the hyperelliptic involution on each curve.

As for stable curves, there is a natural isomorphism $\Phi \colon \mathrm{Pic}(\overline{H}_g) \to \mathrm{Pic}(\overline{\mathcal{H}}_g)$, where $\overline{\mathcal{H}}_g$ is the moduli stack of stable, hyperelliptic curves of genus g. As pointed out on page 469 of [3], since the universal deformation space of a stable, hyperelliptic curve within hyperelliptic curves is smooth, for each $i = 0, \ldots, [(g-1)/2]$ the divisor Ξ_i induces a divisor class ξ_i in $\mathrm{Pic}(\overline{\mathcal{H}}_g)$. Denote also by ξ_i the corresponding rational divisor class in $\overline{H}_g$ under the isomorphism Φ. Though improperly, we shall also denote by $\lambda, \delta_0, \ldots, \delta_{[g/2]}$ the pullbacks to $\overline{H}_g$ or $\overline{\mathcal{H}}_g$ of the corresponding classes in $\overline{M}_g$ or $\overline{\mathcal{M}}_g$. Then relation (3) is simply a restatement of the identity (4.6) on page 469 of [3]. Relation (3) is also recovered in Section 4 below.

4. Reduction

The natural map $\overline{M}_{0,2g+2} \to \overline{H}_g$ is finite and surjective. So, the pullback homomorphism,

$$\mathrm{Pic}(\overline{H}_g) \otimes \mathbb{Q} \to \mathrm{Pic}(\overline{M}_{0,2g+2}) \otimes \mathbb{Q},$$

is injective. Since $\overline{M}_{0,2g+2}$ is smooth and projective, there is a smooth curve $T \subseteq \overline{M}_{2g+2}$ such that the left-hand sides of (2) and (3) pull back to zero on $\overline{M}_{0,2g+2}$ only if they pull back to zero on T. Since $\overline{M}_{0,2g+2}$ is a fine moduli space, there are a family R/T of rational, nodal curves and sections $\zeta_1, \dots, \zeta_{2g+2}$ of the smooth locus of R/T representing the inclusion $T \to \overline{M}_{0,2g+2}$. The curve T may be chosen "general enough" so that R is smooth and each singular fiber of R/T is uninodal.

Let T_0 (resp. T_1, resp. T_2) be the subset of points $t \in T$ for which R_t is singular and one of its irreducible components contains two (resp. an odd number, resp. an even number) of the points $\zeta_1(t), \dots, \zeta_{2g+2}(t)$. We can construct a smooth curve S and a finite covering $\gamma\colon S \to T$ of degree 2 ramified along T_1 but unramified along $T_0 \cup T_2$; see Lecture 3 in [7] for the theory of cyclic coverings. Put $Q := R \times_T S$ and $S_i := \gamma^{-1}(T_i)$ for each $i = 0, 1, 2$. Let B be the blow-up of Q along the nodes of the fibers Q_s for all $s \in S_1$. Then B is smooth. For each $s \in S_1$, let E_s be the rational curve on B contracting to a point on the fiber Q_s. For each $i = 1, \dots, 2g+2$, as ζ_i is a section of Y/T through its smooth locus, $\zeta_i \times 1_S\colon S \to Q$ factors through the blow-up map $B \to Q$; let τ_i denote this section of B/S. Note that $\tau_i(S) \cap E_s = \emptyset$ for each $i = 1, \dots, 2g+2$ and each $s \in S_1$. So the divisor D defined below is reduced.

$$D := \sum_{i=0}^{2g+2} \tau_i(S) + \sum_{s \in S_1} E_s.$$

Moreover, D is smooth, and its restriction to each irreducible component of each fiber of B/S has even degree. So, we can construct a smooth surface M and a finite covering $M \to B$ of degree 2 ramified along D. For each $s \in S_1$, let F_s be the rational curve on M lying over E_s. Let $\widetilde{C}$ be the surface obtained from M by contracting F_s for all $s \in S_1$, and $\widetilde{\pi}\colon \widetilde{C} \to S$ the induced map. Since F_s has self-intersection -1, the surface $\widetilde{C}$ is smooth. Finally, let C be the surface obtained from $\widetilde{C}$ by contracting all rational curves contained in the fibers over S_0. Then the induced map $\pi\colon C \to S$ is a family of stable, hyperelliptic curves representing the composition $\beta\colon S \to T \to \overline{M}_{0,2g+2} \to \overline{H}_g$.

For each $i = 1, \dots, 2g+2$, the section τ_i factors through the covering $M \to B$ because the latter is ramified along $\tau_i(S)$; let σ_i denote this section of M/S. Then σ_i induces a section of $\widetilde{\pi}$; let Σ_i denote the image of this section in $\widetilde{C}$. It follows from our construction that Σ_i is contained in the smooth locus of $\widetilde{C}/S$.

Put $\Sigma := \Sigma_1$ and $\mathcal{L} := \mathcal{O}_{\widetilde{C}}(2\Sigma)$. Let $\overline{W} := \Sigma_1 + \cdots + \Sigma_{2g+2}$. Then $\overline{W}$ is flat over S, and intersects each smooth fiber of $\widetilde{\pi}$ in the ramification scheme of the complete linear system of sections of the restriction of $\mathcal{L}$ to that fiber.

From now on to the end of the article, fix π and $\widetilde{\pi}$ as above. Since $\gamma\colon S \to T$ is finite and surjective, the pullback map $\gamma^*\colon \mathrm{Pic}(T) \otimes \mathbb{Q} \to \mathrm{Pic}(S) \otimes \mathbb{Q}$ is injective. Hence, (2) holds only if (4) holds, whereas (3) holds only if

$$\delta_{0,\pi} - \xi_{0,\pi} - 2(\xi_{1,\pi} + \cdots + \xi_{[(g-1)/2],\pi}) = 0. \tag{8}$$

For each $i = 0, \dots, [(g-1)/2]$ and each $j = 1, \dots, [g/2]$, denote $\Xi_{i,\pi} := \beta^{-1}(\Xi_i)$ and $\Delta_{j,\pi} := \beta^{-1}(\Theta_j)$ with their reduced induced structures. Let

$$\Delta_\pi := \Delta_{1,\pi} \cup \cdots \cup \Delta_{[g/2],\pi} \quad \text{and} \quad \Xi_\pi := \Xi_{0,\pi} \cup \cdots \cup \Xi_{[(g-1)/2],\pi}.$$

Put $\Gamma_\pi := \Delta_\pi \cup \Xi_\pi$. Then Γ_π consists of the points $s \in S$ for which the fiber C_s is singular. By construction, C is smooth everywhere but at the nodes of the fibers of π over $\Xi_{0,\pi}$, where C has singularities of type A_2. It follows that

$$\xi_{0,\pi} = 2[\Xi_{0,\pi}] \quad \text{and} \quad \xi_{i,\pi} = [\Xi_{i,\pi}] \text{ for } i = 1, \dots, [(g-1)/2], \tag{9}$$

$$\delta_{0,\pi} = 2[\Xi_\pi] \quad \text{and} \quad \delta_{j,\pi} = [\Delta_{j,\pi}] \text{ for } j = 1, \dots, [g/2]. \tag{10}$$

Relation (8) follows immediately from the above expressions. Hence (3) holds.

5. Limit linear systems and ramification divisors

5.1. *Limit linear systems.* For each $s \in \Gamma_\pi$, let X_s and Y_s be the irreducible components of $\widetilde{C}_s$, chosen such that $\Sigma_s \subseteq X_s$. Let a_s and b_s be the genera of X_s and Y_s, respectively.

By [8] or [18], there are unique (mod $\mathrm{Pic}(S)$) invertible sheaves $\mathcal{I}$ and $\mathcal{K}$ on $\widetilde{C}$ which agree with $\mathcal{L}$ on the generic fiber of $\widetilde{\pi}$ and satisfy the properties below for each $s \in \Gamma_\pi$:

1. The natural maps $\widetilde{\pi}_*\mathcal{I}(s) \to H^0(\mathcal{I}|X_s)$ and $\widetilde{\pi}_*\mathcal{K}(s) \to H^0(\mathcal{K}|Y_s)$ are injective.
2. The natural maps $\widetilde{\pi}_*\mathcal{I}(s) \to H^0(\mathcal{I}|Y_s)$ and $\widetilde{\pi}_*\mathcal{K}(s) \to H^0(\mathcal{K}|X_s)$ are non-zero.

The sheaves $\widetilde{\pi}_*\mathcal{I}$ and $\widetilde{\pi}_*\mathcal{K}$ are generically of rank 2. Moreover, they are torsion-free because $\widetilde{\pi}$ is flat. So, they are locally free of rank 2 because S is smooth.

Let $s \in \Gamma_\pi$. By Properties 1 and 2 above, the restriction of $\mathcal{I}$ to either X_s or Y_s has non-negative degree. Suppose first that X_s is not rational. Then $\deg \mathcal{I}|X_s \geq 2$ because $\dim \widetilde{\pi}_*\mathcal{I}(s) = 2$. Now, $\mathcal{I}$ has total degree 2, as it agrees with $\mathcal{L}$ on the generic fiber of $\widetilde{\pi}$. It follows that $\deg \mathcal{I}|X_s = 2$ and $\deg \mathcal{I}|Y_s = 0$.

Now, suppose X_s is rational. Then Y_s is not rational, and hence $\deg \mathcal{K}|X_s = 0$ by analogy with the case above. Since $\mathcal{I}$ and $\mathcal{K}$ agree on the generic fiber of $\widetilde{\pi}$, there is an integer ℓ such that $\mathcal{I}$ agrees with $\mathcal{K}(\ell Y_s)$ on a neighborhood of $\widetilde{C}_s$. Since $\deg \mathcal{K}|X_s = 0$, it follows from Property 1 that $\ell > 0$. Now, since X_s is rational, we have $s \in \Xi_{0,\pi}$ and, in particular, $X_s \cdot Y_s = 2$. So $\deg \mathcal{I}|X_s \geq 2$ because $\ell > 0$. As before, $\deg \mathcal{I}|X_s = 2$ and $\deg \mathcal{I}|Y_s = 0$.

By analogy, $\deg \mathcal{K}|X_s = 0$ and $\deg \mathcal{K}|Y_s = 2$.

Since $\mathcal{L}$ and $\mathcal{I}$ agree on the generic fiber of $\widetilde{\pi}$, and restrict to sheaves of equal degree on each irreducible component of each singular fiber, $\mathcal{L}$ and $\mathcal{I}$ agree mod $\mathrm{Pic}(S)$. We may assume that $\mathcal{I} = \mathcal{L}$. By the same reason, we may assume that

$$\mathcal{K} = \mathcal{L}(-\sum_{s \in \Xi_\pi} Y_s - 2\sum_{s \in \Delta_\pi} Y_s). \tag{11}$$

If $s \in S$ is such that C_s is smooth, then $\mathcal{L}_s^{g-1} = \omega_{\widetilde{\pi},s}$. Now, for each $s \in \Gamma_\pi$,

$$\deg \omega_{\widetilde{\pi}}|Y_s = \begin{cases} 2b_s & \text{if } s \in \Xi_\pi, \\ 2b_s - 1 & \text{if } s \in \Delta_\pi. \end{cases}$$

Since $\deg \mathcal{L}|Y_s = 0$, by considerations of degree,

$$\mathcal{L}^{g-1} = \omega_{\widetilde{\pi}}(\sum_{s \in \Xi_\pi} b_s Y_s + \sum_{s \in \Delta_\pi} (2b_s - 1) Y_s) \quad (\mathrm{mod}\ \mathrm{Pic}(S)). \tag{12}$$

5.2. *Ramification divisors.* Let W be the ramification divisor associated to $\mathcal{I}$ and W' that associated to $\mathcal{K}$. In other words, W and W' are the degeneracy schemes of the natural maps $\widetilde{\pi}^*\widetilde{\pi}_*\mathcal{I} \to J^1_{\widetilde{\pi}}(\mathcal{I})$ and $\widetilde{\pi}^*\widetilde{\pi}_*\mathcal{K} \to J^1_{\widetilde{\pi}}(\mathcal{K})$, respectively. Then $\overline{W}$ agrees with W and W' away from the singular fibers of $\widetilde{\pi}$. Moreover, by Property 1 of Subsection 5.1, neither X_s is contained in the support of W, nor Y_s is contained in the support of W'.

Proposition 5.3. *The following expression holds:*

$$\overline{W} = W - \sum_{s\in\Xi_\pi} Y_s - 2\sum_{s\in\Delta_\pi} Y_s. \tag{13}$$

Proof. Let

$$\mathcal{J} := \mathcal{I}(-\sum_{s\in\Gamma_\pi} Y_s). \tag{14}$$

Let w, $w^\dagger$ and w' be the respective determinants of the natural maps,

$$\widetilde{\pi}^*\widetilde{\pi}_*\mathcal{I} \longrightarrow J^1_{\widetilde{\pi}}(\mathcal{I}), \quad \widetilde{\pi}^*\widetilde{\pi}_*\mathcal{J} \longrightarrow J^1_{\widetilde{\pi}}(\mathcal{J}) \quad \text{and} \quad \widetilde{\pi}^*\widetilde{\pi}_*\mathcal{K} \longrightarrow J^1_{\widetilde{\pi}}(\mathcal{K}).$$

The zero scheme of w is W and that of w' is W'. Let $W^\dagger$ be the zero scheme of $w^\dagger$.

As observed in Subsection 5.1, we have $\mathcal{I} = \mathcal{L}$. By (11) and (14), there is a sequence of inclusions $\mathcal{K} \hookrightarrow \mathcal{J} \hookrightarrow \mathcal{I}$, which induces the commutative diagram below.

$$\begin{array}{ccccc}
\widetilde{\pi}^*\det\widetilde{\pi}_*\mathcal{K} & \longrightarrow & \widetilde{\pi}^*\det\widetilde{\pi}_*\mathcal{J} & \longrightarrow & \widetilde{\pi}^*\det\widetilde{\pi}_*\mathcal{I} \\
{\scriptstyle w'}\Big\downarrow & & {\scriptstyle w^\dagger}\Big\downarrow & & {\scriptstyle w}\Big\downarrow \\
\omega_{\widetilde{\pi}} \otimes \mathcal{K}^2 & \xrightarrow{2\sum_{s\in\Delta_\pi} Y_s} & \omega_{\widetilde{\pi}} \otimes \mathcal{J}^2 & \xrightarrow{2\sum_{s\in\Gamma_\pi} Y_s} & \omega_{\widetilde{\pi}} \otimes \mathcal{I}^2.
\end{array}$$

The horizontal maps above are injective, so we will view them as inclusions. By Property 2 of Subsection 5.1, there is a local section of $\widetilde{\pi}_*\mathcal{I}$ that is not zero on Y_s. Now, $\widetilde{\pi}_*\mathcal{I}$ has rank 2. Since $\deg \mathcal{I}|Y_s = 0$, there is also a local section of $\widetilde{\pi}_*\mathcal{I}$ that vanishes on Y_s. So

$$\det\widetilde{\pi}_*\mathcal{J} = (\det\widetilde{\pi}_*\mathcal{I})(-\Gamma_\pi). \tag{15}$$

Now, by (11) and (14),

$$\mathcal{J} \otimes \widetilde{\pi}^*\mathcal{O}_S(-\Delta_\pi) = \mathcal{K}(-\sum_{s\in\Delta_\pi} X_s).$$

Hence, by analogy with (15),

$$\det\widetilde{\pi}_*\mathcal{K} = (\det\widetilde{\pi}_*\mathcal{J})(-\Delta_\pi). \tag{16}$$

Since the two squares in the above diagram are commutative, (15) and (16) imply

$$W + \sum_{s\in\Gamma_\pi} X_s = W^\dagger + \sum_{s\in\Gamma_\pi} Y_s \quad \text{and} \quad W^\dagger + \sum_{s\in\Delta_\pi} X_s = W' + \sum_{s\in\Delta_\pi} Y_s,$$

and hence

$$W - \sum_{s\in\Gamma_\pi} Y_s - \sum_{s\in\Delta_\pi} Y_s = W' - \sum_{s\in\Gamma_\pi} X_s - \sum_{s\in\Delta_\pi} X_s \tag{17}$$

as divisors on $\widetilde{C}$. Both sides of (17) are effective because X_s and Y_s are distinct for each $s \in \Gamma_\pi$. Moreover, since X_s is not in the support of W and Y_s is not in the support of W' for each $s \in \Gamma_\pi$, neither X_s nor Y_s is in the support of either side of (17). So the right-hand side of (13) is a relative Cartier divisor on $\widetilde{C}/S$. As both sides of (13) are relative Cartier divisors and agree on the generic fiber of $\widetilde{\pi}$, they agree everywhere. $\square$

6. Proof of the theorem

6.1. *Preliminaries.* Let $D\overline{w}\colon \mathcal{O}_{\widetilde{C}} \to J^1_{\widetilde{\pi}}(\mathcal{O}_{\widetilde{C}}(\overline{W}))$ be the derivative of the section $\overline{w}$ of $\mathcal{O}_{\widetilde{C}}(\overline{W})$ given by $\overline{W}$. Since $\overline{W}$ is the disjoint union of subschemes isomorphic to S, the zero scheme Z of $D\overline{w}$ is empty. Since Z has the expected dimension, $[Z] = c_2(J^1_{\widetilde{\pi}}(\mathcal{O}_{\widetilde{C}}(\overline{W})))$. From exact sequence (5) with $\widetilde{\pi}$ for π and $\mathcal{O}_{\widetilde{C}}(\overline{W})$ for $\mathcal{L}$ we get

$$c_2(J^1_{\widetilde{\pi}}(\mathcal{O}_{\widetilde{C}}(\overline{W}))) = c_1(\mathcal{O}_{\widetilde{C}}(\overline{W}))c_1(\omega_{\widetilde{\pi}} \otimes \mathcal{O}_{\widetilde{C}}(\overline{W})).$$

Let $K := c_1(\omega_{\widetilde{\pi}})$. Since W is the zero scheme of a map $\widetilde{\pi}^* \det \widetilde{\pi}_*\mathcal{L} \to \omega_{\widetilde{\pi}} \otimes \mathcal{L}^2$, we have $W = K + 2c_1(\mathcal{L}) - \widetilde{\pi}^*c_1(\widetilde{\pi}_*\mathcal{L})$. From (13) we get

$$\overline{W} = K + D - \widetilde{\pi}^*c_1(\widetilde{\pi}_*\mathcal{L}), \quad \text{where} \quad D := 2c_1(\mathcal{L}) - \sum_{s\in\Xi_\pi} Y_s - 2\sum_{s\in\Delta_\pi} Y_s. \tag{18}$$

By (12), there is an invertible sheaf N on S such that

$$(g-1)c_1(\mathcal{L}) = K + \sum_{s\in\Xi_\pi} b_sY_s + \sum_{s\in\Delta_\pi}(2b_s-1)Y_s + \widetilde{\pi}^*c_1(N). \tag{19}$$

6.2. *Computing* $c_1(\widetilde{\pi}_*\mathcal{L})$. By definition, $\mathcal{L} = \mathcal{O}_{\widetilde{C}}(2\Sigma)$. We claim that $(\widetilde{\pi}_*\mathcal{L}(s), \mathcal{L}_s)$ has no base points for any $s \in S$. This is well-known for $s \notin \Gamma_\pi$. Let now $s \in \Gamma_\pi$. Since $\overline{M}_{0,2g+2}$ parameterizes *stable* $2g+2$-pointed rational curves, there is an integer $i > 1$ such that $\Sigma_{i,s} \subseteq X_s$. As $\mathcal{O}_{\widetilde{C}}(2\Sigma_i)$ and $\mathcal{L}$ agree on the generic fiber of $\widetilde{\pi}$, and restrict to sheaves of equal degree on the irreducible components of $\widetilde{C}_s$, they agree on a neighborhood of $\widetilde{C}_s$. So $2\Sigma_s$ and $2\Sigma_{i,s}$ are divisors of the linear system $(\widetilde{\pi}_*\mathcal{L}(s), \mathcal{L}_s)$, which has therefore no base points.

It follows from our claim above that the natural sequence below is exact.

$$0 \to \widetilde{\pi}_*\mathcal{O}_{\widetilde{C}}(\Sigma) \to \widetilde{\pi}_*\mathcal{O}_{\widetilde{C}}(2\Sigma) \to \widetilde{\pi}_*\mathcal{O}_{\widetilde{C}}(2\Sigma)|\Sigma \to 0.$$

So $\widetilde{\pi}_*\mathcal{O}_{\widetilde{C}}(\Sigma)$ is invertible on S. Now, the natural inclusion $\mathcal{O}_{\widetilde{C}} \to \mathcal{O}_{\widetilde{C}}(\Sigma)$ pushes down to a map $\mathcal{O}_S \to \widetilde{\pi}_*\mathcal{O}_{\widetilde{C}}(\Sigma)$ whose restriction to each point of S is injective. Since $\widetilde{\pi}_*\mathcal{O}_{\widetilde{C}}(\Sigma)$ is invertible, $\mathcal{O}_S = \widetilde{\pi}_*\mathcal{O}_{\widetilde{C}}(\Sigma)$. So, it follows from the above exact sequence that

$$c_1(\widetilde{\pi}_*\mathcal{L}) = c_1(\widetilde{\pi}_*\mathcal{O}_{\widetilde{C}}(2\Sigma)|\Sigma) = \widetilde{\pi}_*(2\Sigma^2).$$

Since $c_1(\mathcal{L}) = 2\Sigma$, it follows that

$$\widetilde{\pi}_*(c_1(\mathcal{L})^2) = 2c_1(\widetilde{\pi}_*\mathcal{L}). \tag{20}$$

6.3. *Conclusion.* For each $s \in \Gamma_\pi$, we have

$$\widetilde{\pi}_*(KY_s) = \begin{cases} 2b_s[s] & \text{if } s \in \Xi_\pi, \\ (2b_s-1)[s] & \text{if } s \in \Delta_\pi. \end{cases} \tag{21}$$

Now, since $X_s + Y_s = \widetilde{\pi}^*[s]$, using the projection formula we get

$$\widetilde{\pi}_* Y_s^2 = \begin{cases} -2[s] & \text{if } s \in \Xi_\pi, \\ -[s] & \text{if } s \in \Delta_\pi. \end{cases} \tag{22}$$

Let $\kappa_{1,\widetilde{\pi}} := \widetilde{\pi}_* K^2$. Since $[Z] = \overline{W}(K + \overline{W})$, it follows from (18) that

$$[Z] = (K + D - \widetilde{\pi}^* c_1(\widetilde{\pi}_*\mathcal{L}))(2K + D - \widetilde{\pi}^* c_1(\widetilde{\pi}_*\mathcal{L})). \tag{23}$$

Expanding, we get

$$[Z] = 2K^2 + 3KD + D^2 - 3K\widetilde{\pi}^* c_1(\widetilde{\pi}_*\mathcal{L}) - 2D\widetilde{\pi}^* c_1(\widetilde{\pi}_*\mathcal{L}) + (\widetilde{\pi}^* c_1(\widetilde{\pi}_*\mathcal{L}))^2.$$

Pushing down to S, and using the projection formula, we get

$$\widetilde{\pi}_*[Z] = 2\kappa_{1,\widetilde{\pi}} + \widetilde{\pi}_*(3KD + D^2) - 6(g-1)c_1(\widetilde{\pi}_*\mathcal{L}) - 2\widetilde{\pi}_* D c_1(\widetilde{\pi}_*\mathcal{L}).$$

Now, $c_1(\mathcal{L})Y_s = 0$ for every $s \in \Gamma_\pi$, because $\Sigma_s \subseteq X_s$. Thus, using the expression for D in (18), we get

$$\begin{aligned}\widetilde{\pi}_*[Z] =& 2\kappa_{1,\widetilde{\pi}} + 6\widetilde{\pi}_*(Kc_1(\mathcal{L})) - 6(g-1)c_1(\widetilde{\pi}_*\mathcal{L}) + 4\widetilde{\pi}_*(c_1(\mathcal{L})^2) - 8c_1(\widetilde{\pi}_*\mathcal{L}) \\ & - \sum_{s\in\Xi_\pi} \widetilde{\pi}_*(3KY_s - Y_s^2) - \sum_{s\in\Delta_\pi} \widetilde{\pi}_*(6KY_s - 4Y_s^2).\end{aligned}$$

Using (20), (21) and (22), the right-hand side of the above expression becomes

$$2\kappa_{1,\widetilde{\pi}} + 6\widetilde{\pi}_*(Kc_1(\mathcal{L})) - 3(g-1)\widetilde{\pi}_*(c_1(\mathcal{L})^2) - \sum_{s\in\Xi_\pi}(6b_s + 2)[s] - \sum_{s\in\Delta_\pi}(12b_s - 2)[s].$$

Using (19), we obtain

$$\widetilde{\pi}_*[Z] = 2\kappa_{1,\widetilde{\pi}} + 3\widetilde{\pi}_*(Kc_1(\mathcal{L})) - 6c_1(N) - \sum_{s\in\Xi_\pi}(6b_s + 2)[s] - \sum_{s\in\Delta_\pi}(12b_s - 2)[s].$$

Multiplying by $g-1$, and using (19) again, the right-hand side of the above equation becomes

$$(2g+1)\kappa_{1,\widetilde{\pi}} + \sum_{x\in\Xi_\pi}(6b_s(b_s - g + 1) - 2(g-1))[s] + \sum_{s\in\Delta_\pi}(12b_s(b_s - g) + 2g + 1)[s].$$

Since $a_s + b_s = g - 1$ if $s \in \Xi_\pi$ and $a_s + b_s = g$ if $s \in \Delta_\pi$, we get

$$(g-1)\widetilde{\pi}_*[Z] = (2g+1)\kappa_{1,\widetilde{\pi}} - \sum_{s\in\Xi_\pi}(6a_sb_s + 2(g-1))[s] - \sum_{s\in\Delta_\pi}(12a_sb_s - 2g - 1)[s].$$

Now, by Grothendieck-Riemann-Roch,

$$\kappa_{1,\widetilde{\pi}} = 12\lambda_{\widetilde{\pi}} - 2\sum_{s\in\Xi_\pi}[s] - \sum_{s\in\Delta_\pi}[s].$$

Moreover, since $\widetilde{\pi}_*\omega_{\widetilde{\pi}} = \pi_*\omega_\pi$, we have $\lambda_{\widetilde{\pi}} = \lambda_\pi$. Thus

$$\begin{aligned}(g-1)\widetilde{\pi}_*[Z] =& 12(2g+1)\lambda_\pi - 6\sum_{x\in\Xi_\pi}(a_sb_s + g)[s] - 12\sum_{s\in\Delta_\pi} a_sb_s[s] \\ =& 12(2g+1)\lambda_\pi - 6\sum_{i=0}^{[(g-1)/2]}(i+1)(g-i)[\Xi_{i,\pi}] - 12\sum_{j=1}^{[g/2]} j(g-j)[\Delta_{j,\pi}].\end{aligned}$$

Using (9) and (10) we get

$$(g-1)\widetilde{\pi}_*[Z] = 12(2g+1)\lambda_\pi - 3g\xi_{0,\pi} - 6\sum_{i=1}^{[(g-1)/2]}(i+1)(g-i)\xi_{\pi,i} - 12\sum_{j=1}^{[g/2]} j(g-j)\delta_{\pi,j}.$$

Then $(g-1)\widetilde{\pi}_*[Z]$ is a multiple of the left-hand side of (4). Since $\widetilde{\pi}_*[Z]=0$, we get (4), thus proving Cornalba's and Harris' relation (2).

7. A variant

There is yet another way to obtain relation (2), still inspired by Faber's general philosophy of writing down the class of the empty set. We present here just a rough account of this different approach, as we intend to give details in a forthcoming article.

Keep the notation used so far in the article. The particular empty set we will consider is the closure $\overline{R}_{1,2}$ of the locus $R_{1,2} \subseteq \widetilde{C}\times_S \widetilde{C}$ of pairs (P,Q) of points of a smooth fiber $\widetilde{C}_s$ of $\widetilde{\pi}$ for which $\mathcal{O}_{\widetilde{C}_s}(P+2Q) \cong \mathcal{L}|\widetilde{C}_s$. This set is clearly empty, as $\mathcal{L}$ has relative degree 2 over S. On the other hand, we can express the class of $\overline{R}_{1,2}$ in $A^2(\widetilde{C}\times_S\widetilde{C})$ as the product,

$$(L_1+L_2-\Delta-\rho^*c_1(\widetilde{\pi}_*\mathcal{L})-F)(K_2+L_1+L_2-\Delta-\rho^*c_1(\widetilde{\pi}_*\mathcal{L})-F), \tag{24}$$

where $\rho\colon \widetilde{C}\times_S\widetilde{C}\to S$ is the natural map, $K_i := p_i^*c_1(\omega_\pi)$ and $L_i := p_i^*c_1(\mathcal{L})$ for $i=1,2$, where p_1 and p_2 are the projections of $\widetilde{C}\times_S\widetilde{C}$ onto the indicated factors, Δ is the diagonal of $\widetilde{C}\times_S\widetilde{C}$ and F is a correction term supported in the singular fibers of ρ. The computation of F is similar to that which leads to relation (13).

Being (24) zero, we obtain a relation in $A^2(\widetilde{C}\times_S\widetilde{C})$. By reason of dimension, this relation does not yield immediately a relation in $\mathrm{Pic}(S)$. However, using Faber's recipe in [10], "once zero always zero", we multiply (24) by divisor classes on $\widetilde{C}\times_S\widetilde{C}$, and then push down the ensuing products to S in order to get relations in $\mathrm{Pic}(S)$. There are two natural divisor classes on $\widetilde{C}\times_S\widetilde{C}$: the diagonal Δ and K_1. Hence we get two relations in $\mathrm{Pic}(S)$:

$$\rho_*((L_1+L_2-\Delta-\rho^*c_1(\widetilde{\pi}_*\mathcal{L})-F)(K_2+L_1+L_2-\Delta-\rho^*c_1(\widetilde{\pi}_*\mathcal{L})-F)\Delta)=0, \tag{25}$$

$$\rho_*((L_1+L_2-\Delta-\rho^*c_1(\widetilde{\pi}_*\mathcal{L})-F)(K_2+L_1+L_2-\Delta-\rho^*c_1(\widetilde{\pi}_*\mathcal{L})-F)K_1)=0. \tag{26}$$

The left-hand side of (25) is simply $\widetilde{\pi}_*[Z]$, for $[Z]$ expressed as in (23). We use (26) to find an expression for $c_1(\widetilde{\pi}_*\mathcal{L})$ which, when used in (25), yields (4) after a few computations, as in Subsection 6.3.

The main difficulty in the above alternative proof of (2) is that $\widetilde{C}\times_S\widetilde{C}$ is not smooth. More precisely, Δ and the irreducible components of the singular fibers of ρ are singular in $\widetilde{C}\times_S\widetilde{C}$, and hence equations (24)-(26) don't make immediate sense. This difficulty can be overcome because the singular locus of $\widetilde{C}\times_S\widetilde{C}$ is finite, of codimension 3.

References

[1] E. Arbarello – M. Cornalba, *The Picard groups of the moduli spaces of curves*, Topology **26** (1987), 153–171.

[2] M. Boggi – M. Pikaart, *Galois covers of moduli of curves*, Compositio Math. **120**, no. 2 (2000), 171–191.

[3] M. Cornalba – J. Harris, *Divisor classes associated to families of stable varieties, with applications to the moduli space of curves*, Ann. Sci. Ec. Norm. Super. **21** (1988), 455–475.
[4] F. Cukierman, *Families of Weierstrass points*, Duke Math. J. **58**, no. 2 (1989), 317–346.
[5] P. Deligne – D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHES **36** (1969), 75–109.
[6] S. Diaz, "Exceptional Weierstrass points and the divisor on moduli space that they define", Mem. Amer. Math. Soc., vol. 56, no. 327, Amer. Math. Soc., Providence, 1985.
[7] H. Esnault – E. Viehweg, "Lectures on vanishing theorems", DMV Seminar, Band 20, Birkhäuser, Basel, 1992.
[8] E. Esteves, *Wronski algebra systems on families of singular curves*, Ann. Sci. Ec. Norm. Super. Ser. IV, **29** (1996), 107–134.
[9] E. Esteves, *Linear systems and ramification points on reducible nodal curves*, In: "Algebra Meeting", A. Garcia – E. Esteves – A. Pacheco (eds.), Mat. Contemp. **14**, Soc. Bras. Mat., Rio de Janeiro, 1998, pp. 21–35.
[10] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*, In: "The Dutch intercity seminar on moduli", C. Faber – E. Looijenga (eds.), Friedr. Vieweg & Sohn, Braunschweig, 1999, pp. 109–129.
[11] L. Gatto, *On the closure in $\overline{M}_g$ of smooth curves having a special Weierstrass point*, Math. Scand., to appear.
[12] L. Gatto – F. Ponza, *Derivatives of Wronskians with applications to families of special Weierstrass points*, Trans. Amer. Math. Soc. **351**, no. 6 (1999), 2233–2255.
[13] J. Harer, *The second homology group of the mapping class group of an orientable surface*, Invent. Math. **72** (1983), 221–239.
[14] J. Harris – I. Morrison, "Moduli of curves", Graduate Texts in Mathematics 187, Springer, New York, 1998.
[15] E. Looijenga, *Smooth Deligne-Mumford compactifications by means of Prym level structures*, J. Algebraic Geom. **3**, no. 2 (1994), 283–293.
[16] D. Mumford, *Stability of projective varieties*, Enseign. Math. **24** (1977), 39–110.
[17] D. Mumford, *Towards an enumerative geometry of the moduli space of curves*, In: "Arithmetic and Geometry, Vol. II", M. Artin – J. Tate (eds.), Progress in Math., vol. 36, Birkhäuser, Boston, 1983, pp. 271–328.
[18] Z. Ran, *Degenerations of linear systems*, Unpublished, 1985.

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